

k -wise Erdős-Ko-Rado theorems: Stability Analysis and New Generalizations

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Abstract

We consider the following generalization of the seminal Erdős-Ko-Rado theorem, due to Frankl [6]. For some $k \geq 2$, let \mathcal{F} be a k -wise intersecting family of r -subsets of an n element set X , i.e. for $F_1, \dots, F_k \in \mathcal{F}$, $\cap_{i=1}^k F_i \neq \emptyset$. If $r \leq \frac{(k-1)n}{k}$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. We prove a stability version of this theorem, analogous to similar results of Dinur-Friedgut, Keevash-Mubayi and others for the Erdős-Ko-Rado theorem. The technique we use is a generalization of Katona's circle method, initially employed by Keevash, which uses expansion properties of a particular Cayley graph of the symmetric group.

Next, we extend Frankl's theorem in a graph-theoretic direction. For a graph G , and $r \geq 1$, let $\mathcal{I}^r(G)$ denote the set of all independent vertex sets of size r . Similarly, let $\mathcal{M}^r(G)$ denote the family of all vertex sets of size r containing a maximum independent set, and let $\mathcal{H}^r(G) = \mathcal{I}^r(G) \cup \mathcal{M}^r(G)$. We will consider k -wise intersecting families in $\mathcal{H}^r(M_n)$, where M_n is a perfect matching on $2n$ vertices, and prove an analog of Frankl's theorem. This theorem can also be considered as a k -wise version of a theorem of Bollobás and Leader [2].

Key words. intersection theorems, stability, matchings.

1 Introduction

For a positive integer n , let $[n] = \{1, 2, \dots, n\}$. For positive integers i and j with $i \leq j$, let $[i, j] = \{i, i+1, \dots, j\}$ ($[i, j] = \emptyset$ if $i > j$). Similarly let $(i, j] = \{i+1, \dots, j\}$, which is empty if $i+1 > j$. The notations (i, j) and $[i, j)$ are similarly defined. Let $\binom{[n]}{r}$ be the family of all r -subsets of $[n]$. For $\mathcal{F} \subseteq \binom{[n]}{r}$ and $v \in [n]$, let $\mathcal{F}(v) = \{F \in \mathcal{F} : v \in F\}$, called a *star* in \mathcal{F} , centered at v . Call $\mathcal{F} \subset \binom{[n]}{r}$ k -wise intersecting if for any $F_1, \dots, F_k \in \mathcal{F}$, $\bigcap_{i=1}^k F_i \neq \emptyset$. Frankl [6] proved the following theorem for k -wise intersecting families.

Theorem 1.1 (Frankl). *Let $\mathcal{F} \subset \binom{[n]}{r}$ be k -wise intersecting. If $r \leq \frac{(k-1)n}{k}$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$.*

It is trivial to note that the $k = 2$ case of Theorem 1.1 is the seminal Erdős-Ko-Rado theorem [5].

Theorem 1.2 (Erdős-Ko-Rado). *Let $\mathcal{F} \subseteq \binom{[n]}{r}$ be intersecting. If $r \leq n/2$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$.*

1.1 Stability

The classical extremal problem is to determine the maximum size and structure of a family on a given ground set of size n which avoids a given forbidden configuration \mathcal{F} . For example, the Erdős-Ko-Rado theorem finds the maximum size of a set system on the set $[n]$, which does not have a pair of disjoint subsets. Often, only a few trivial structures attain this extremal number. In case of the EKR theorem, the only extremal structure when $r < \frac{n}{2}$ is that of a star in $\binom{[n]}{r}$. A natural further step is to ask whether non-extremal families which have size close to the extremal number also have structure similar to any of the extremal structures. This approach was first pioneered by Simonovits [11], to answer a question in extremal graph theory, and a similar notion for set systems was recently formulated by Mubayi [10]. Apart from being an interesting question in its own right, this approach has found many applications, especially in extremal hypergraph theory, where exact results are typically much harder to prove.

One of the first stability results in extremal set theory was the theorem of Hilton and Milner [7] which proved a stability result for the Erdős-Ko-Rado theorem by giving an upper bound on the maximum size of non-star intersecting families. Other stability results for the Erdős-Ko-Rado theorem have been recently proved by Dinur-Friedgut [4], Keevash [8], Keevash-Mubayi [9] and others. We prove the following stability result for Theorem 1.1.

Theorem 1.3. *For some $k \geq 2$, let $1 < r < \frac{(k-1)n}{k}$, and let $\mathcal{F} \subseteq \binom{[n]}{r}$ be a k -wise intersecting family. Then for any $0 \leq \epsilon < 1$, there exists a $0 \leq \delta < 1$ (in particular, $\delta \leq \frac{\epsilon}{rn^4}$ suffices) such that if $|\mathcal{F}| \geq (1 - \delta)\binom{n-1}{r-1}$, then there is an element $v \in [n]$ such that $|\mathcal{F}(v)| \geq (1 - \epsilon/n)\binom{n-1}{r-1}$.*

We note that for $k \geq 2$, \mathcal{F} is k -wise intersecting implies that it is intersecting. Hence if $r < n/2$, the results obtained in the papers mentioned above suffice as stability results for Theorem 1.1. Consequently, the main interest of our theorem is in the structural information that it provides when $n/2 \leq r < (k-1)n/k$.

1.2 Matchings

Next, we consider a graph-theoretic generalization of Theorem 1.1. For a graph G (with vertex set and edge set denoted by $V(G)$ and $E(G)$ respectively) and $r \geq 1$, let $\mathcal{I}^r(G)$ denote the set of all independent vertex sets of size r . Let $\mathcal{M}^r(G)$ denote the family of all vertex sets of size r containing a maximum independent set and let $\mathcal{H}^r(G) = \mathcal{I}^r(G) \cup \mathcal{M}^r(G)$. For a vertex $x \in V(G)$, let $\mathcal{H}_x^r(G) = \{A \in \mathcal{H}^r(G) : x \in A\}$. Define $\mathcal{I}_x^r(G)$ and $\mathcal{M}_x^r(G)$ in a similar manner. Henceforth we will consider the perfect matching graph on $2n$ vertices (and n edges), and denote it by M_n . We will consider k -wise intersecting families in $\mathcal{H}^r(M_n)$, and prove the following analog of Frankl's theorem.

Theorem 1.4. *For $k \geq 2$, let $r \leq \frac{(k-1)(2n)}{k}$, and let $\mathcal{F} \subseteq \mathcal{H}^r(M_n)$ be k -wise intersecting. Then,*

$$|\mathcal{F}| \leq \begin{cases} 2^{r-1} \binom{n-1}{r-1} & \text{if } r \leq n \\ 2^{2n-r} \binom{n-1}{r-n-1} + 2^{2n-r-1} \binom{n-1}{r-n} & \text{otherwise} \end{cases}$$

If $r < \frac{(k-1)(2n)}{k}$, then equality holds if and only if $\mathcal{F} = \mathcal{H}_x^r(M_n)^1$ for some $x \in V(M_n)$.

It is not hard to observe that the $k = 2$ case of Theorem 1.4 is the following theorem of Bollobás and Leader [2].

Corollary 1.5 (Bollobás-Leader). *Let $1 \leq r \leq n$, and let $\mathcal{F} \subseteq \mathcal{I}^r(M_n)$ be an intersecting family. Then, $|\mathcal{F}| \leq 2^{r-1} \binom{n-1}{r-1}$. If $r < n$, equality holds if and only if $\mathcal{F} = \mathcal{I}_x^r(M_n)$ for some $x \in V(M_n)$.*

Note that if $r < n$, then $\mathcal{H}^r(M_n) = \mathcal{I}^r(M_n)$ and $\mathcal{M}^r(M_n) = \emptyset$. Similarly, if $r > n$, $\mathcal{H}^r(M_n) = \mathcal{M}^r(M_n)$ and $\mathcal{I}^r(M_n) = \emptyset$. In the case $r = n$, we have $\mathcal{H}^r(M_n) = \mathcal{I}^r(M_n) = \mathcal{M}^r(M_n)$. We also observe that the main interest of our theorem is in the case $r > n$ for the bound, and $r \geq n$ for the characterization of the extremal structures. This is because of the previously stated fact that if a family \mathcal{F} is k -wise intersecting ($k \geq 2$), it is also intersecting.

The rest of the paper is organized as follows: in Section 2, we give a proof of Theorem 1.3 and in Section 3, we prove Theorem 1.4.

¹ $|\mathcal{H}_x^r(M_n)| = 2^{2n-r} \binom{n-1}{r-n-1} + 2^{2n-r-1} \binom{n-1}{r-n}$, when $r > n$.

2 Proof of Theorem 1.3

Suppose $\mathcal{F} \subseteq \binom{[n]}{r}$ be a k -wise intersecting family, with $r < \frac{(k-1)n}{k}$. For any $0 \leq \epsilon < 1$, let $\delta = \frac{\epsilon}{rn^4}$ and suppose $|\mathcal{F}| \geq (1 - \delta)\binom{n-1}{r-1}$. We will show that \mathcal{F} contains a large star.

2.1 Some Lemmas

In this section, we will prove some Katona-type lemmas which we will employ later in the proof of the main theorem. We introduce some notation first. Consider a permutation $\sigma \in S_n$ as a sequence $(\sigma(1), \dots, \sigma(n))$. We say that two permutations μ and π are *equivalent* if there is some $i \in [n]$ such that $\pi(x) = \mu(x+i)$ for all $x \in [n]$.² Let P_n be the set of equivalence classes, called *cyclic* orders on $[n]$. For a cyclic order σ and some $x \in [n]$, call the set $\{\sigma(x), \dots, \sigma(x+r-1)\}$ a σ -interval (called just an interval, if there is no ambiguity) *starting* at x , *ending* in $x+r-1$, and *containing* the points $(x, x+1, \dots, x+r-1)$ (addition again mod n). The following lemma is due to Frankl [6]. We include the short proof below as we will build on these ideas in the proofs of the other lemmas.

Lemma 2.1 (Frankl). *Let $\sigma \in P_n$ be a cyclic order on $[n]$, and \mathcal{F} be a k -wise intersecting family of intervals of length $r \leq (k-1)n/k$. Then, $|\mathcal{F}| \leq r$.*

Proof. Let $\mathcal{F}^c = \{[n] \setminus F : F \in \mathcal{F}\}$. Let $|\mathcal{F}| = |\mathcal{F}^c| = m$. We will prove that $m \leq r$. Since $r \leq (k-1)n/k$, we have $n \leq k(n-r)$. Suppose $G_1, \dots, G_k \in \mathcal{F}^c$. Clearly $\cup_{i=1}^k G_i \neq [n]$; otherwise $\cap_{i=1}^k ([n] \setminus G_i) = \emptyset$, which is a contradiction. Let $G \in \mathcal{F}^c$. Without loss of generality, suppose G ends in n . We now assign indices from $[1, k(n-r)]$ to sets in \mathcal{F}^c . For every set $G' \in \mathcal{F}^c \setminus \{G\}$, assign the index x to G' if G' ends in x . Assign all indices in $[n, k(n-r)]$ for G . Consider the set of indices $[k(n-r)]$ and partition them into equivalence classes mod $n-r$. Suppose there is an equivalence class such that all k indices in that class are assigned. Let $\{H_i\}_{i \in [k]}$ be the k sets in \mathcal{F}^c which end at the k indices in the equivalence class. It is easy to note that $\cup_{i=1}^k H_i = [n]$, which is a contradiction. So for every equivalence class, there exists an index which has not been assigned to any set in \mathcal{F}^c . This implies that there are at least $n-r$ indices in $[k(n-r)]$ which are unassigned. Each set in $\mathcal{F}^c \setminus \{G\}$ has one index assigned to it, and G has $k(n-r) - n + 1$ indices assigned to it. This gives us $m-1 + k(n-r) - n + 1 + n-r \leq k(n-r)$, which simplifies to $m \leq r$, completing the proof. ◇

We will now characterize the case when $|\mathcal{F}| = r$, in the following lemma.

Lemma 2.2. *Let $\sigma \in P_n$ be a cyclic order on $[n]$, and let \mathcal{F} be a k -wise intersecting family of intervals of length $r < (k-1)n/k$. If $|\mathcal{F}| = r$, then \mathcal{F} consists of all intervals which contain a point x .*

²Addition is carried out mod n , so $x+i$ is either $x+i$ or $x+i-n$, depending on which lies in $[n]$.

Proof. As in the proof of Lemma 2.1, we consider \mathcal{F}^c and assume (without loss of generality) that there exists $F \in \mathcal{F}^c$ which ends in n . It is clear from the proof of Lemma 2.1 that if $|\mathcal{F}| = r$, there are exactly $n - r$ indices in $[k(n - r)]$, one from each equivalence class, which are not assigned to any set in \mathcal{F}^c . Since F ends in n , all indices in $[n, k(n - r)]$ (and there will be at least 2) will be assigned. It will be sufficient to show that the set of unassigned indices is an interval of length $n - r$ that starts at some $x \in [r]$, as this would imply that every set in \mathcal{F} contains x .

Let x be the smallest unassigned index in $[n - 1]$. Clearly $x \leq r$. Let $x \equiv j \pmod{n - r}$. We will show that $x + i$ is unassigned for each $0 \leq i \leq n - r - 1$. We argue by induction on i , with the base case being $i = 0$. Let $y = x + i$ for some $1 \leq i \leq n - r - 1$. Suppose y is assigned and Y is the set in \mathcal{F}^c that ends in y . We know by the induction hypothesis that $y - 1$ is unassigned, so every other index in the same equivalence class as $y - 1$ is assigned. Call this equivalence class E_{y-1} . Consider all indices in E_{y-1} which lie in $(y - 1, n]$ and let I_1 be the set of these indices, all assigned. Similarly, consider all indices in E_{y-1} which lie in $[1, y - 1)$ and call this set I_2 . Let $I'_2 = \{j + 1 : j \in I_2\}$. I'_2 contains indices in the same equivalence class as y , and are assigned (since they are all less than x , and x is the smallest unassigned index). Let $J = I_1 \cup I'_2$, and since J contains only assigned indices, let \mathcal{H} be the subfamily of \mathcal{F}^c to which indices in J are assigned. Let p be the largest index in I_1 and let q be the smallest index in I'_2 . Since $n < k(n - r)$, the set which ends in q contains $p + 1$. The family $\mathcal{H} \cup \{Y\}$ has at most k sets, and the union of all sets in this family is $[n]$. This is a contradiction. Thus y is unassigned. \diamond

Now, let $\mathcal{F} \subseteq \binom{[n]}{r}$ be a k -wise intersecting family for some $r < \frac{(k - 1)n}{k}$. For each cyclic order $\sigma \in P_n$, let \mathcal{F}_σ be the sets in \mathcal{F} that are intervals in σ . We say that σ is *saturated* if $|\mathcal{F}_\sigma| = r$; otherwise call it *unsaturated*. By Lemma 2.2, if σ is saturated, all sets in \mathcal{F}_σ contain a common point, say v , so call σ *v-saturated* to identify the common point.

For $i \leq n$, define an *adjacent transposition* A_i on a cyclic order σ as an operation that swaps the elements in positions i and $i + 1$ ($i + 1 = 1$ if $i = n$) of σ . We are now ready to prove our next lemma.

Lemma 2.3. *For a k -wise intersecting family $\mathcal{F} \subseteq \binom{[n]}{r}$ with $r < \frac{(k - 1)n}{k}$, let $\sigma \in P_n$ be a v -saturated cyclic order. Let μ be the cyclic order obtained from σ by an adjacent transposition A_i , $i \in [n] \setminus \{v, v - 1\}$ ($v - 1 = n$ if $v = 1$). If μ is saturated, then it is v -saturated.*

Proof. Without loss of generality (relabeling if necessary), assume σ is n -saturated, so $1 \leq i \leq n - 2$. As before, we consider the families of complements \mathcal{F}^c and observe that the interval which begins at n and ends in $n - r - 1$ contains all the $n - r$ unassigned indices.

- Suppose $i \in (n - r - 1, n - 1)$. In μ , let A be the interval that begins at $i + 1$ and ends in some j . Clearly, $j \neq n - r - 1$. Let μ be saturated.

Suppose first that $j \in (n - r - 1, n)$. Then the interval of unassigned indices in σ is also unassigned in μ , so μ is n -saturated. Next, we argue that if $j \in [1, n - r - 1]$, j cannot be an assigned index. This is because all the indices in the set $\{n\} \cup [1, j) \cup (j, n - r - 1]$ are unassigned in μ , and by Lemma 2.2, all the unassigned indices in a saturated order occur in an interval of length $n - r$. So assume $j = n$ and suppose j is assigned. By Lemma 2.2, the index $n - r$ will be unassigned, which is only possible if $i = n - r$ (otherwise, the μ -interval of length $n - r$ which ends at index $n - r$ would be the same as the σ -interval which ends at $n - r$). This implies that $n = 2(n - r)$, implying $k \geq 3$. Now consider the following intervals, all of which are sets in \mathcal{F}^c : the one starting at 1 and ending at $i = n - r$ in σ , the one starting at $i = n - r$ and ending at $n - 1$ in σ and the one starting at $i + 1 = n - r + 1$ and ending at n in μ . The union of these three sets is $[n]$, a contradiction.

- Suppose $i = n - r - 1$. Like before, the only possibilities to consider are when either n or $n - r - 1$ are assigned indices in μ . Suppose n is assigned in μ . This means that $i + 1 = n - r$ is unassigned in μ , by Lemma 2.2. However, the μ -interval which ends at $i + 1$ is the same as the σ -interval that ends at $i + 1$ (except the different order of the elements in the interval), which is a contradiction. So suppose $n - r - 1$ is assigned in μ . By Lemma 2.2, $n - 1$ is unassigned in μ . This is only possible if the interval ending in $n - 1$ starts at $i + 1$. This means $n = 2(n - r)$ and an argument identical to Case 1 suffices.
- Suppose $i \in [1, n - r - 1]$. Now the μ -interval of length $n - r$ ending in $n - r - 1$ is the same as the σ -interval, so $n - r - 1$ is unassigned in μ . Hence assume n is assigned in μ . Then the μ -interval starting at $i + 1$ ends in n and is a set in \mathcal{F}^c . Clearly, the union of this set with the σ -interval starting at 1 and ending in $n - r$, which is also a set in \mathcal{F}^c , is $[n]$, a contradiction.

◇

2.2 Cayley Graphs

In this small section, we gather some facts about expansion properties of a specific Cayley graph of the symmetric group. We will consider the Cayley graph G on S_{n-1} generated by the set of adjacent transpositions $A = \{(12), \dots, (n - 2, n - 1)\}$. In particular, the vertex set of G is S_{n-1} and two permutations σ and μ are adjacent if $\mu = \sigma \circ a$, for some $a \in A$. We note that the transposition operates by exchanging adjacent positions (as opposed to consecutive values). G is an $n - 2$ -regular graph. It was shown by Keevash [8], using a result of Bacher [1], that G is an α -expander, for $\alpha > \frac{1}{n^3}$, i.e. for any $H \subseteq V(G)$ with $|H| \leq \frac{|V(G)|}{2}$, we have $N(H) \geq \frac{|H|}{n^3}$, where $N(H)$ is the set of all vertices in $V(G) \setminus H$ which are adjacent to some vertex in H .

2.3 Proof of Main Theorem

Proof of Theorem 1.3. We will finish the proof of Theorem 1.3 in this section. We can identify every cyclic order in P_n with a permutation $\sigma \in S_n$ having $\sigma(n) = n$. Restricting σ to $[n-1]$ gives a bijection between P_n and S_{n-1} . Let U be the set of unsaturated cyclic orders in P_n . We have

$$\begin{aligned} r!(n-r)!|\mathcal{F}| &= \sum_{\sigma \in P_n} |\mathcal{F}_\sigma| \\ &\leq \sum_{\sigma \in P_n} r - |U| \\ &= r(n-1)! - |U|. \end{aligned}$$

This gives us $|U| \leq r(n-1)! - r!(n-r)!(1-\delta)\binom{n-1}{r-1} = r\delta(n-1)!$, implying that there are at least $(1-r\delta)(n-1)!$ saturated orders in P_n .

We now consider the Cayley graph G defined above, with the vertex set being P_n and the generating set being the set of adjacent transpositions $A = \{(12), \dots, (n-2, n-1)\}$. Suppose S is a subset of saturated cyclic orders. Since G is an $\frac{1}{n^3}$ -expander, if $n^3 r \delta \leq \frac{|S|}{(n-1)!} \leq \frac{1}{2}$, we get $N(S) > |S|/n^3 \geq r\delta(n-1)!$. This means that there is a saturated cyclic order in $N(S)$. We will use this observation to show that the subgraph of G induced by the set of all saturated cyclic orders, say H , has a large component. Consider the set of all components in H . Call a component small if it has at most $n^3 r \delta(n-1)!$ saturated orders. We argue that the total size of all small components is at most $n^3 r \delta(n-1)!$. Suppose not. Let S' be the union of (at least 2) small components such that $n^3 r \delta(n-1)! \leq |S'| \leq 2n^3 r \delta(n-1)! \leq (n-1)!/2$. The last inequality follows because $\delta < \frac{1}{rn^4}$. Now using the above observation, $N_H(S')$ is non-empty, a contradiction. Thus there is a component of size at least $(1 - n^3 r \delta)(n-1)!$. Call this component H' . By Lemma 2.3, there exists a $v \in V(G)$ such that every cyclic order in H' is v -saturated. Thus, $r!(n-r)!|\mathcal{F}(v)| \geq \sum_{\sigma \in H'} |\mathcal{F}_\sigma| \geq r(1 - n^3 r \delta)(n-1)!$, giving $|\mathcal{F}(v)| \geq (1 - \epsilon/n)\binom{n-1}{r-1}$, as $\delta = \frac{\epsilon}{rn^4}$. \square

3 Proof of Theorem 1.4

Let $V(M_n) = \{1, 2, \dots, 2n\}$, and let $E(M_n) = \{\{1, n+1\}, \{2, n+2\}, \dots, \{n, 2n\}\}$. Call two vertices which share an edge as *partners*. We consider cyclic orderings of the set $V(G)$, i.e. a bijection between $V(G)$ and $[2n]$ with certain properties. In particular, call a cyclic ordering of $V(G)$ *good* if all partners are exactly n apart in the cyclic order. More formally, if c is a bijection from $V(G)$ to $[2n]$, c is a good cyclic ordering if for any $i \in [n]$, $c(i) = c(i+n) + n$ (modulo $2n$, so $c(i) = c(i+n) - n$ if $c(i+n) > n$). It is fairly simple to note that the total number of good cyclic orderings, regarding cyclically equivalent orderings as identical, is $2^{n-1}(n-1)!$. Every interval in a good cyclic ordering will be either an independent set in M_n (if $r \leq n$) or contain a maximum independent set (if

$r > n$). Now let $\mathcal{F} \subseteq \mathcal{H}^r(G)$ be k -wise intersecting for $r \leq \frac{(k-1)(2n)}{k}$. Using an argument identical to the proof of Lemma 2.1, we can conclude that for any good cyclic ordering c , there can be at most r sets in \mathcal{F} that are intervals in c . Similarly using Lemma 2.2, we get that if there are exactly r sets in \mathcal{F} that are intervals in c , all r sets must contain a specific point $v \in [2n]$. For a given set $F \in \mathcal{F}$, in how many good cyclic orderings is it an interval? The answer depends on the value of r . Suppose $r \leq n$. In this case, F is an interval in $r!(n-r)!2^{n-r}$ good cyclic orderings. Thus we have $|\mathcal{F}|r!(n-r)!2^{n-r} \leq r(n-1)!2^{n-1}$, giving $|\mathcal{F}| \leq \binom{n-1}{r-1}2^{r-1}$. Note that this bound also follows directly from Corollary 1.5, since k -wise intersecting implies intersecting, and $r \leq n$ implies that $\mathcal{H}^r(M_n) = \mathcal{I}^r(G)$. Now suppose $r > n$. Then $I^r(G) = \emptyset$ and $\mathcal{H}^r(G) = \mathcal{M}^r(G)$. We can think of each set in \mathcal{F} as containing both vertices from $r-n$ edges, and exactly 1 vertex each from the remaining $2n-r$ edges. Hence the number of good cyclic orders in which a set $F \in \mathcal{F}$ is contained is $(2n-r)!(r-n)!2^{r-n}$. This gives us the following inequality.

$$\begin{aligned}
|\mathcal{F}| &\leq \frac{r(n-1)!2^{n-1}}{(2n-r)!(r-n)!2^{r-n}} \\
&= \frac{n(n-1)!2^{2n-r-1}}{(2n-r)!(r-n)!} + \frac{(r-n)(n-1)!2^{2n-r-1}}{(2n-r)!(r-n)!} \\
&= \binom{n}{r-n}2^{2n-r-1} + \binom{n-1}{r-n-1}2^{2n-r-1} \\
&= \binom{n-1}{r-n-1}2^{2n-r-1} + \binom{n-1}{r-n}2^{2n-r-1} + \binom{n-1}{r-n-1}2^{2n-r-1} \\
&= 2^{2n-r} \binom{n-1}{r-n-1} + 2^{2n-r-1} \binom{n-1}{r-n}.
\end{aligned}$$

This completes the proof of the bound. We will now prove that the extremal families are essentially unique. To simplify the argument, and because Corollary 1.5 suffices when $r < n$, we let $r \geq n$, so $2n-r \leq n$. Now suppose that $r < \frac{(k-1)(2n)}{k}$ and $|\mathcal{F}| = 2^{2n-r} \binom{n-1}{r-n-1} + 2^{2n-r-1} \binom{n-1}{r-n}$. Then for each good cyclic ordering c , there are exactly r sets from \mathcal{F} that are intervals in c . Using Lemma 2.2, we can conclude that each good cyclic ordering is saturated.

Consider the good cyclic ordering π defined by $\pi(i) = i$ for $1 \leq i \leq 2n$ and assume without loss of generality that it is $2n$ -saturated. Since the number of good cyclic orderings are $2^{n-1}(n-1)!$, we will identify all good cyclic orderings with bijections σ from $[2n]$ to itself that satisfy $\sigma(n) = n$ and $\sigma(2n) = 2n$.

For $1 \leq i \leq n-2$, define an adjacent transposition A_i on $\sigma \in S_{n-1}$ as the operation that swaps the elements in positions i and $i+1$ of σ . For each permutation $p \in S_{n-1}$, define the following good cyclic ordering σ on $[2n]$: for $1 \leq i \leq n-1$, let $\sigma(i) = p(i)$ and for $n+1 \leq i \leq 2n-1$, let $\sigma(i) = p(i-n) + n$. Also let $\sigma(i) = i$ if $i \in \{n, 2n\}$. Denote the set of good cyclic orders obtained from permutations in S_{n-1} in this manner by C_{n-1} . Now for $1 \leq i \leq n-2$, define an analogous adjacent transposition T_i for any good cyclic ordering σ as an operation that swaps the elements in positions i and $i+1$ and also the

elements in positions $i + n$ and $i + n + 1$ of σ , so the resulting cyclic ordering, say μ , is also a good cyclic ordering. Note also that if $\sigma \in C_{n-1}$, then $\mu \in C_{n-1}$. We now prove a lemma that is similar to Lemma 2.3. The proof will be very similar to that of Lemma 2.3, so we will omit many of the details.

Lemma 3.1. *For a k -wise intersecting family $\mathcal{F} \subseteq \mathcal{H}^r(M_n)$, with $r < \frac{(k-1)(2n)}{k}$, let σ be a $2n$ -saturated good cyclic ordering. Let μ be the good cyclic order obtained from σ by an adjacent transposition T_i , $i \in [n-2]$. If μ is saturated, then it is $2n$ -saturated.*

Proof. As in Lemmas 2.1 and 2.2, we again consider compliments of sets of \mathcal{F} that are intervals in σ . By Lemma 2.2, we know that the interval starting at $2n$ and ending at $2n - r - 1$ contains all of the $2n - r$ unassigned indices. Now, suppose T_i is an adjacent transposition, for $1 \leq i \leq n - 2$. Recall that T_i swaps elements in position i and $i + 1$, and also the elements in positions $i + n$ and $i + n + 1$. Suppose μ , obtained from σ by S_i is saturated, but not $2n$ -saturated. We consider the following cases.

- Suppose $i = 2n - r - 1$. In this case, $2n$ cannot be an assigned index in μ since that would mean $2n - r$ is unassigned in μ . This would be a contradiction because the interval ending at $2n - r$ is the same in μ and σ . So suppose $2n - r - 1$ is assigned, implying that $2n - 1$ is unassigned in μ . This means that the interval beginning at $i + n + 1 = 3n - r$ ends in $2n - 1$, giving $3n = 2r$ (and hence, $2n - r = n/2$). This yields $k \geq 5$. Now consider the following sets: the μ -interval ending in $n/2 - 1$, the four σ -intervals ending in $n/2$, n , $3n/2$ and $2n - 1$. All of these indices are assigned in the respective good cyclic orderings and the union of these 5 sets is $[2n]$, a contradiction.
- Suppose $i \in [1, 2n - r - 1]$. Now all intervals which end at some point the interval $[2n - r - 1, n]$, and there are at least 2, will be unchanged in μ , so $2n - r - 1$ is an unassigned index in μ and all indices in $(2n - r - 1, n]$ are assigned. This implies by Lemma 2.2 that the set of unassigned indices is the same in μ , as required.
- Suppose $i \in (2n - r - 1, n - 1)$. Here the interval ending in $2n - r - 1$ is the same in both σ and μ , so suppose $2n$ is assigned in μ . This means that $2n - r$ is unassigned in μ , implying $i = 2n - r$. Since $2n$ is assigned in μ , we have $(i + n + 1) + (2n - r - 1) = 2n$, which yields $i = r - n$. Hence $3n = 2r$ and $k \geq 5$. Now consider the following sets: the μ -interval ending in $2n$, the four σ -intervals ending in $n/2$, n , $3n/2$ and $2n - 1$. All of these indices are assigned in the respective good cyclic orderings and the union of the 5 sets is $[2n]$, a contradiction.

◇

Now for $1 \leq i \leq n$, define a *swap* operation W_i on a good cyclic ordering σ as an operation that exchanges the elements in positions i and $n + i$ of σ . We will now prove the following lemma about the swap operation.

Lemma 3.2. *For a k -wise intersecting family $\mathcal{F} \subseteq \mathcal{H}^r(M_n)$ with $n < r < \frac{(k-1)(2n)}{k}$, let σ be a $2n$ -saturated good cyclic ordering. Let μ be the good cyclic order obtained from σ by the swap W_{n-1} . If μ is saturated, then it is $2n$ -saturated.*

Proof. We first observe that $n < r$ implies $k \geq 3$. We consider two cases for the proof. As before, we have the interval beginning in $2n$ and ending in $2n - r - 1$ containing all of the $2n - r$ unassigned indices.

- Suppose $r = n + 1$, so $2n - r = n - 1$. Now $2n - r - 1$ is still unassigned in μ . This implies that $2n$ is assigned in μ and $2n - r = n - 1$ is unassigned. Now consider the following three intervals of length $2n - r$, identified by their end points: the σ -interval ending in $n - 1$, which is an assigned index in σ and the μ -intervals ending in $n + 1$ and $2n$, both of which are assigned indices in μ . All 3 sets lie in \mathcal{F}^c , and their union is $[2n]$, a contradiction.³
- Suppose $n - 1 > 2n - r$. Now the intervals of length $2n - r$ ending at the points in the interval $[2n - r - 1, n - 1]$ (which has length at least 2) are the same in both σ and μ . In other words, $2n - r - 1$ is unassigned in μ and all the other indices in the interval are assigned. This means that the set of unassigned indices is the same in μ , as required.

◇

We will now consider two cases, $r = n$ and $r > n$, since the proofs for the two cases are slightly different. Suppose first that $r > n$. Since every good cyclic ordering is saturated (and since we have assumed that π is $2n$ -saturated), we can use Lemmas 3.1 and 3.2 to infer that every good cyclic ordering is $2n$ -saturated. To finish the proof of this case, we will show that each set in $\mathcal{H}_{2n}^r(M_n)$ is an interval in some such good cyclic ordering. Let $A \in \mathcal{H}_{2n}^r(M_n)$. Then A contains $r - n$ edges (i.e. both vertices in $r - n$ edges) and $2n - r$ other vertices, one each from the other $2n - r$ edges. Suppose first that $n \in A$, so A contains the edge $\{n, 2n\}$. Let the other $r - n - 1$ edges be $\{\{x_1, y_1\}, \dots, \{x_{r-n-1}, y_{r-n-1}\}\}$, with each $x_i \in [n - 1]$ and each $y_i \in [n + 1, 2n - 1]$. Let $L = \{l_1, \dots, l_{2n-r}\}$ be the set of the remaining $2n - r$ vertices in A . We now construct a good cyclic ordering σ in which A is an interval. To define σ , it clearly suffices to define values of $\sigma(i)$ for $1 \leq i \leq n - 1$. So for $1 \leq i \leq r - n - 1$, let $\sigma(i) = x_i$, and for $1 \leq i \leq 2n - r$, let $\sigma(i + r - n - 1) = l_i$. Here the σ -interval of length r , ending at $r - 1$, is precisely A . Now suppose that $n \notin A$. Let the $r - n$ edges be $\{\{x_1, y_1\}, \dots, \{x_{r-n}, y_{r-n}\}\}$ and let $L = \{l_1, \dots, l_{2n-r}\}$ be the remaining $2n - r$ vertices. A good cyclic ordering σ in which A is an interval can be constructed as follows: for $1 \leq i \leq 2n - r$, let $\sigma(i) = l_i$ and for $2n - r + 1 \leq i \leq n - 1$, let $\sigma(i) = x_{i-(2n-r)}$. In this case, the σ -interval of length r ending at $n - 1$, is A .

For $r = n$, we observe by Lemma 3.1 that every good cyclic ordering in C_{n-1} is $2n$ -saturated. Again, we will show that every set in $\mathcal{H}_{2n}^r(M_n)$ is an interval

³Strictly speaking, this argument requires $n \geq 4$. However, when $n \leq 3$ and $r = n + 1$, we have $k \geq 4$ and a simple ad hoc argument suffices.

in some $\sigma \in C_{n-1}$. Let $A \in \mathcal{H}_{2n}^r(M_n)$. Note that A is a maximum independent set in M_n and contains no edges. Let $V = A \cap [n-1]$, $|V| = s$, for some $s \leq r$ and let $W = A \setminus \{V \cup \{2n\}\}$. Let $V = \{v_1, \dots, v_s\}$ and $W = \{w_1, \dots, w_{r-1-s}\}$. Construct a good cyclic ordering $\sigma \in C_{n-1}$ as follows: for $1 \leq i \leq s$, define $\sigma(i) = v_i$, and for $s+1 \leq i \leq r-1$, set $\sigma(i) = w_{i-s} - n$. Then the σ -interval of length r , ending at s , is A . This completes the proof of the theorem. \square

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